

Global spectra, polytopes and stacky invariants

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Abstract

Given a convex polytope, we define its geometric spectrum, a stacky version of Batyrev's stringy E-functions, and we prove a stacky version of a formula of Libgober and Wood about the E-polynomial of a smooth projective variety. As an application, we get a closed formula for the variance of the geometric spectrum and a Noether's formula for two dimensional Fano polytopes (polytopes whose vertices are primitive lattice points; a Fano polytope is not necessarily smooth). We also show that this geometric spectrum is equal to the algebraic spectrum (the spectrum at infinity of a tame Laurent polynomial whose Newton polytope is the polytope alluded to). This gives an explanation and some positive answers to Hertling's conjecture about the variance of the spectrum of tame regular functions.

1 Introduction

Let X be a smooth projective variety of dimension n with Hodge numbers $h^{p,q}(X)$. It follows from Hirzebruch-Riemann-Roch theorem that

$$\frac{d^2}{du^2} E(X; u, 1)|_{u=1} = \frac{n(3n-5)}{12} c_n(X) + \frac{1}{6} c_1(X) c_{n-1}(X) \quad (1)$$

where $E(X; u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q$, see Libgober and Wood [20] and also Borisov [6, Proposition 2.2]. By duality, we get

$$\sum_{p,q} (-1)^{p+q} h^{p,q}(X) (p - \frac{n}{2})^2 = \frac{n}{12} c_n(X) + \frac{1}{6} c_1(X) c_{n-1}(X) \quad (2)$$

If X is more generally a n -dimensional projective variety with at most log-terminal singularities (we will focus in this paper on the toric case), Batyrev [2] has proved a stringy version of formula (1)

$$\frac{d^2}{du^2} E_{st}(X; u, 1)|_{u=1} = \frac{n(3n-5)}{12} e_{st}(X) + \frac{1}{6} c_{st}^{1,n}(X) \quad (3)$$

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where E_{st} is the stringy E -function of X , e_{st} is the stringy Euler number and $c_{st}^{1,n}(X)$ is a stringy version of $c_1(X)c_{n-1}(X)$.

On the singularity theory side, the expected mirror partners of toric varieties are the Givental-Hori-Vafa models [15], [18], in general a class of Laurent polynomial. One associates to such functions their *spectrum at infinity*, a sequence $\alpha_1, \dots, \alpha_\mu$ of rational numbers, suitable logarithms of the eigenvalues of the monodromy at infinity of the function involved (see [24]; the main features are recalled in section 5). A specification of mirror symmetry is that the spectrum at infinity of a given Givental-Hori-Vafa model is related to the degrees of the (orbifold) cohomology groups of its mirror variety (orbifold). So one can expect a formula similar to (2) involving the spectrum at infinity of any tame regular function: the aim of this text is to look for such a counterpart. The key observation is that the spectrum at infinity of a Laurent polynomial can be described (under a tameness condition due to Kouchnirenko [19], see section 5) with the help of the Newton filtration of its Newton polytope. Since a polytope determines a stacky fan [4], we are led to define a “stacky” version of the E -polynomial. Given a Laurent polynomial f with (simplicial) Newton polytope P , global Milnor number μ and spectrum at infinity $\alpha_1, \dots, \alpha_\mu$, the program is thus as follows:

- to construct a stacky version of the E -polynomial, the *geometric spectrum* of P : we define $Spec_P^{geo}(z) := (z-1)^n \sum_{v \in N} z^{-\nu(v)}$ where ν is the Newton function of the polytope P , see section 4. This geometric spectrum is closely related to the Ehrhart series and δ -vector of the polytope P , more precisely to their twisted versions studied by Stapledon [25] and Mustata-Payne [21]; this function is also an orbifold Poincaré series (see corollary 4.2.4), thanks to the description of orbifold cohomology given by Borisov, Chen and Smith [4, Proposition 4.7],
- to show that this geometric spectrum is equal to the (generating function of the) spectrum at infinity of f and this is done by showing that both functions are Hilbert-Poincaré series of isomorphic graded rings (see corollary 6.1.2); this gives the expected identification between the spectrum at infinity and orbifold degrees,
- to show a formula

$$\frac{d^2}{dz^2} Spec_P^{geo}(z)|_{z=1} = \frac{n(3n-5)}{12} \mu + \frac{1}{6} \hat{\mu}$$

where $\hat{\mu}$ is a linear combination of intersection numbers: this is done using the analog of Batyrev’s stringy formula (3), see theorem 7.1.3.

At the end we get a version of (2) for the spectrum at infinity of Laurent polynomials, see theorem 7.1.5:

$$\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \frac{n}{12} \mu + \frac{1}{6} \hat{\mu} \quad (4)$$

It should be emphasized that this formula is in essence produced by Hirzebruch-Riemann-Roch theorem.

In order to enlighten formula (4), assume that $N = \mathbb{Z}^2$ and that P is a full dimensional reflexive lattice polytope in $N_{\mathbb{R}}$. Then we have the following well-known Noether’s formula

$$12 = \mu_P + \mu_{P^\circ} \quad (5)$$

where P° is the polar polytope of P and μ_P (*resp.* μ_{P°) is the normalized volume of P (*resp.* P°), see equation (11) (by Pick’s formula, $\mu_P = \text{Card}(\partial P \cap N)$ if P is reflexive). We show in section 8

that if P is a Fano lattice polytope (a polytope is Fano if its vertices are primitive lattice points) we have $\widehat{\mu}_P = \mu_{P^\circ}$. From formula (4), we then get

$$\sum_{i=1}^{\mu} (\alpha_i - 1)^2 = \frac{1}{6} \mu_P + \frac{1}{6} \mu_{P^\circ} \quad (6)$$

which is a generalization of Noether's formula (5): indeed, a reflexive polytope P is Fano and its algebraic and/or geometric spectrum satisfies $\sum_{i=1}^{\mu} (\alpha_i - 1)^2 = 2$ (*version 3 of july 2016*: I am informed that an analogous result has been proved independently by Batyrev and Schaller, see [3]).

Last, it follows from equation (4) that if $\widehat{\mu} \geq 0$ we have

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 \geq \frac{\alpha_{max} - \alpha_{min}}{12} \quad (7)$$

where α_{max} (*resp.* α_{min}) denotes the maximal (*resp.* minimal) spectral value, because $\alpha_{max} - \alpha_{min} = n$ for Laurent polynomials. This inequality is expected to be true for any tame regular function: this is the global version of Hertling's conjecture about the variance of the spectrum, see section 9. For instance, formula (6) show that this will be the case in the two dimensional case if the Newton polytope of f is Fano.

This paper is organized as follows: in section 2 we recall the basic facts on polytopes and toric varieties that we will use. In section 3, we discuss of what should be the spectrum of a polytope. The geometric spectrum is defined in section 4 and the algebraic spectrum is defined in section 5: both are compared in section 6. The previous results are used in section 7 in order to get formula (4). We show Noether's formula (6) for Fano polytopes in section 8. Last, we use our results in order to motivate (and show in some cases) the conjecture about the variance of the algebraic spectrum in section 9.

This text owes much to Batyrev's work [1], [2]. The starting point was [1, Remark 3.13] and its close resemblance with Hertling's conjecture about the variance of the spectrum of an isolated singularity [16]: this link is previously alluded to in [17].

2 Polytopes and toric varieties (framework)

We give in this section an overview of the results that we will use and we set the notations.

2.1 Polytopes and reflexive polytopes

Let N be the lattice \mathbb{Z}^n , M its dual lattice, $\langle \cdot, \cdot \rangle$ the pairing between $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. A full dimensional lattice polytope $P \subset N_{\mathbb{R}}$ is the convex hull of a finite set of N such that $\dim P = n$. If P is a full dimensional lattice polytope containing the origin in its interior, there exists, for each facet (face of dimension $n - 1$) F of P , $u_F \in M_{\mathbb{Q}}$ such that

$$P \subset \{n \in N_{\mathbb{R}}, \langle u_F, n \rangle \leq 1\} \text{ and } F = P \cap \{n \in N_{\mathbb{R}}, \langle u_F, n \rangle = 1\} \quad (8)$$

This gives the hyperplane presentation

$$P = \cap_F \{n \in N_{\mathbb{R}}, \langle u_F, n \rangle \leq 1\} \quad (9)$$

We define, for $v \in N_{\mathbb{R}}$, $\nu_F(v) := \langle u_F, v \rangle$ and $\nu(v) := \max_F \nu_F(v)$ where the maximum is taken over the facets of P .

Definition 2.1.1 *The function $\nu : N_{\mathbb{R}} \rightarrow \mathbb{R}$ is the Newton function of P .*

If P is a full dimensional lattice polytope in $N_{\mathbb{R}}$ containing the origin, the polytope

$$P^\circ = \{m \in M_{\mathbb{R}}, \langle m, n \rangle \leq 1 \text{ for all } n \in P\}$$

is the *polar polytope* of P . The vertices of P° are in correspondence with the facets of P via

$$u_F \text{ vertex of } P^\circ \leftrightarrow F = P \cap \{x \in N_{\mathbb{R}}, \langle u_F, x \rangle = 1\} \quad (10)$$

A lattice polytope P is *reflexive* if it contains the origin and if P° is a lattice polytope.

All the polytopes considered in this paper are full dimensional lattice polytopes containing the origin in their interior. For such a polytope P , we define its *normalized volume*

$$\mu_P := n! \text{vol}(P) \quad (11)$$

where the volume $\text{vol}(P)$ is normalized such that the volume of the unit cube is equal to 1.

2.2 Ehrhart polynomial and Ehrhart series

Let Q be a full dimensional lattice polytope. The function $\ell \mapsto \text{Ehr}_Q(\ell) := \text{Card}((\ell Q) \cap M)$ is a polynomial of degree n , the *Ehrhart polynomial*. We have

$$F_Q(z) := \sum_{m \geq 0} \text{Ehr}_Q(m) z^m = \frac{\delta_0 + \delta_1 z + \cdots + \delta_n z^n}{(1 - z)^{n+1}} \quad (12)$$

where the δ_j 's are positive integers [5, Theorem 3.12]: F_Q is the *Ehrhart series* and the vector

$$\delta = (\delta_0, \dots, \delta_n) \in \mathbb{N}^{n+1} \quad (13)$$

is the δ -vector of the polytope Q . We have

$$\delta_0 = 1, \delta_1 = \text{Card}(Q \cap M) - (n + 1), \delta_n = \text{Card}(\text{Int}(Q) \cap M) \quad (14)$$

and

$$\delta_0 + \cdots + \delta_n = n! \text{vol}(Q) \quad (15)$$

see [5, Chapter 3]. The δ -vector gives a characterization of reflexive polytopes, see for instance [5, Theorem 4.6]: the polytope Q is reflexive if and only if $\delta_i = \delta_{n-i}$ for $i = 0, \dots, n$.

2.3 Toric varieties

Let Δ be a fan in $N_{\mathbb{R}}$ and denote by $\Delta(i)$ the set of its cones of dimension i . The *rays* of Δ are its one-dimensional cones. Let $X := X_\Delta$ be the toric variety of the fan Δ : X is *simplicial* if each cone Δ is generated by independent vectors of $N_{\mathbb{R}}$, *complete* if the support of its fan (the union of its cones) is $N_{\mathbb{R}}$.

One can get toric varieties from polytopes in the following ways: a full dimensional lattice polytope Q in $M_{\mathbb{R}}$ yields a toric variety X_Q , associated with the normal fan Σ_Q of Q , which is a fan in N ; alternatively, if $P \subset N_{\mathbb{R}}$ is a full dimensional lattice polytope containing the origin in

its interior we get a complete fan Δ_P in $N_{\mathbb{R}}$ by taking the cones over the proper faces of P and we will denote by X_{Δ_P} the associated toric variety. Both constructions are dual, see for instance [7, Exercise 2.3.4]: if P° is the polar polytope of the polytope P in $N_{\mathbb{R}}$ then Δ_P is the normal fan of ℓP° where ℓ is an integer such that ℓP° is a lattice polytope and $X_{\Delta_P} = X_{\ell P^\circ}$. In particular, $X_{\Delta_P} = X_{P^\circ}$ if P is reflexive.

Recall that a projective normal toric variety X is *Fano* (*resp. weak Fano*) if the anticanonical divisor $-K_X$ is \mathbb{Q} -Cartier and ample (*resp. nef and big*). The variety X is *Gorenstein* if K_X is a Cartier divisor. (Weak) Fano toric varieties play an important role in our vision of mirror symmetry, see section 5.2. We will say that a full dimensional lattice polytope P containing the origin in its interior is *Fano* if its vertices are primitive lattice points of N , *smooth Fano* if each of its facets has exactly n vertices forming a basis of the lattice N . It should be emphasized that a Fano polytope is not necessarily smooth.

Otherwise stated, all toric varieties that we will consider are complete and simplicial.

2.4 Stacky fans and orbifold cohomology

Let Δ be a complete simplicial fan, ρ_1, \dots, ρ_r be its rays generated respectively by the primitive vectors v_1, \dots, v_r of N . Choose $b_1, \dots, b_r \in N$ whose images in $N_{\mathbb{Q}}$ generate the rays ρ_1, \dots, ρ_r : the data $\Delta = (N, \Delta, \{b_i\})$ is a *stacky fan*, see [4]. In particular, let P be a lattice polytope containing the origin such that $\Delta := \Delta_P$ is simplicial: there are a_i such that $b_i := a_i v_i \in \partial P \cap N$ and we will call the stacky fan $\Delta = (N, \Delta, \{b_i\})$ the *stacky fan of P* . In this situation, we define, for a cone $\sigma \in \Delta$,

- N_σ the subgroup generated by $b_i, \rho_i \subseteq \sigma$,
- $N(\sigma) = N/N_\sigma$,
- the fan Δ/σ in $N(\sigma)_{\mathbb{Q}}$: this is the set $\{\tilde{\tau} = \tau + (N_\sigma)_{\mathbb{Q}}, \sigma \subseteq \tau, \tau \in \Delta\}$
- $\text{Box}(\sigma) := \{\sum_{\rho_i \subseteq \sigma} \lambda_i b_i, \lambda_i \in]0, 1[\}$

One associates to this stacky fan a (separated) Deligne-Mumford stack $\mathcal{X}(\Delta)$, see [4, Proposition 3.2]. We will denote by $H_{orb}^\bullet(\mathcal{X}(\Delta), \mathbb{Q})$ its orbifold cohomology (with rational coefficients) and by $A_{orb}^*(\mathcal{X}(\Delta))$ its orbifold Chow ring (with rational coefficients). By [4, Proposition 4.7] we have

$$H_{orb}^{2i}(\mathcal{X}(\Delta), \mathbb{Q}) = \bigoplus_{\sigma \in \Delta} \bigoplus_{v \in \text{Box}(\sigma) \cap N} H^{2(i-\nu(v))}(X_{\Delta/\sigma}, \mathbb{Q}) \quad (16)$$

where ν is the Newton function of P (see definition 2.1.1).

2.5 Batyrev's stringy functions

Let X_Δ be a normal \mathbb{Q} -Gorenstein toric variety and $\rho : Y \rightarrow X_\Delta$ be a toric (log-)resolution defined by a refinement Δ' of Δ , see for instance [7, Proposition 11.2.4]. The irreducible components of the exceptional divisor of ρ are in one-to-one correspondence with the primitive generators v'_1, \dots, v'_q of the rays of $\Delta'(1)$ of Y that do not belong to $\Delta(1)$ and in the formula

$$K_Y = \rho^* K_{X_\Delta} + \sum_{i=1}^q a_i D_i \quad (17)$$

we have $a_i = \varphi(v'_i) - 1$ where φ is the support function of the divisor K_{X_Δ} , see for instance [7, Lemma 11.4.10]. In our toric situation we have $a_i > -1$ because $\varphi(v'_i) > 0$.

Recall the E -polynomial of a smooth variety X defined by

$$E(X, u, v) := \sum_{p,q=0}^n (-1)^{p+q} h^{p,q}(X) u^p v^q \quad (18)$$

where the $h^{p,q}(X)$'s are the Hodge numbers of X . It is possible to extend this definition to singular spaces having log-terminal singularities and to get *stringy* invariants that extend topological invariants of smooth varieties. Here is the construction: let $\rho : Y \rightarrow X$ be a resolution of $X := X_\Delta$ as above, $I = \{1, \dots, q\}$ and put, for any subset $J \subset I$,

$$D_J := \cap_{j \in J} D_j \text{ if } J \neq \emptyset, \quad D_J := Y \text{ if } J = \emptyset \text{ and } D_J^\circ = D_J - \bigcup_{j \in I-J} D_j$$

The following definition is due to Batyrev [1] (we assume that the product over \emptyset is 1; recall that $a_i > -1$):

Definition 2.5.1 *Let X be a toric variety. The function*

$$E_{st}(X, u, v) := \sum_{J \subset I} E(D_J^\circ, u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \quad (19)$$

is the stringy E -function of X . The number

$$e_{st}(X) := \lim_{u,v \rightarrow 1} E_{st}(X, u, v) \quad (20)$$

is the stringy Euler number.

The stringy E -function can be defined using motivic integrals, see [1] and [26]. By [1, Theorem 3.4], $E_{st}(X, u, v)$ do not depend on the resolution. In our setting, E_{st} depends only on the variable $z := uv$, and we will write $E_{st}(X, z)$ instead of $E_{st}(X, u, v)$.

In section 4.2.2 we will use a modified version of the stringy E -function in order to compute the geometric spectrum of a polytope.

3 The spectrum of a polytope

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$. In this text, a spectrum $Spec_P$ of P is *a priori* an ordered sequence of rational numbers $\alpha_1 \leq \dots \leq \alpha_\mu$ that we will identify with the generating function $Spec_P(z) := \sum_{i=1}^{\mu} z^{\alpha_i}$. The specifications are the following ($d(\alpha_i)$ denotes the multiplicity of α_i in the $Spec_P$):

- *Rationality* : the α_i 's are rational numbers,
- *Positivity* : the α_i 's are positive numbers,
- *Poincaré duality* : $Spec_P(z) = z^n Spec_P(z^{-1})$,

- *Volume* : $\lim_{z \rightarrow 1} \text{Spec}_P(z) = n! \text{vol}(P) =: \mu_P$
- *Normalisation* : $d(\alpha_1) = 1$
- *Modality (Lefschetz)* : $d(\alpha_1) \leq d(\alpha_2) \leq \dots \leq d(\alpha_\ell)$ if $\alpha_\ell \leq [\frac{n}{2}]$

In particular, Spec_P is contained in $[0, n]$ and $\sum_{i=1}^{\mu} \alpha_i = \frac{n}{2} \mu_P$. Basic example: if P is a smooth Fano polytope in \mathbb{R}^n , the Poincaré polynomial $\sum_{i=1}^n b_{2i}(X_{\Delta_P}) z^i$ is a spectrum of P .

4 Geometric spectrum of a polytope

We define here the geometric spectrum of a polytope and we give several methods in order to compute it. It will follow that the geometric spectrum is indeed a spectrum in the sense of the previous section. Recall that the toric varieties considered here are assumed to be simplicial.

4.1 The geometric spectrum

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior. Recall the Newton function ν of P of definition 2.1.1.

Definition 4.1.1 *The function*

$$\text{Spec}_P^{\text{geo}}(z) := (z - 1)^n \sum_{v \in N} z^{-\nu(v)}$$

is the geometric spectrum of the polytope P . The number $e_P := \lim_{z \rightarrow 1} \text{Spec}_P^{\text{geo}}(z)$ is the geometric Euler number of P .

It will follow from proposition 4.2.2 that $\text{Spec}_P^{\text{geo}}(z) = \sum_{i=1}^{e_P} z^{\beta_i}$ for an ordered sequence $\beta_1 \leq \dots \leq \beta_{e_P}$ of non-negative rational numbers. We shall also say that the sequence $\beta_1, \beta_2, \dots, \beta_{e_P}$ is the *geometric spectrum* of the polytope P . We shall also see that e_P is the normalized volume of P .

4.2 Various interpretations

We give three methods to compute $\text{Spec}_P^{\text{geo}}$, showing that it yields finally a spectrum of P in the sense of section 3. The first one and the third one are inspired by the works of Mustata-Payne [21] and Stapledon [25]. The second one is inspired by Batyrev's stringy E -functions.

4.2.1 First interpretation: fundamental domains

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior, $\Delta := \Delta_P$ be the corresponding complete fan as in section 2.3. We identify each vertex of P with an element $b_i \in N$. If $\sigma \in \Delta(r)$ is generated by b_1, \dots, b_r , set

$$\square(\sigma) := \left\{ \sum_{i=1}^r q_i b_i, q_i \in [0, 1[, i = 1, \dots, r \right\},$$

and

$$\text{Box}(\sigma) := \left\{ \sum_{i=1}^r q_i b_i, q_i \in]0, 1[, i = 1, \dots, r \right\}$$

($\text{Box}(\sigma)$ has already been defined in section 2.4).

Lemma 4.2.1 *We have*

$$\text{Spec}_P^{\text{geo}}(z) = \sum_{r=0}^n (z-1)^{n-r} \sum_{\sigma \in \Delta(r)} \sum_{v \in \square(\sigma) \cap N} z^{\nu(v)} \quad (21)$$

and $e_P = n! \text{vol}(P) =: \mu_P$.

Proof. Let $\sigma \in \Delta(r)$. A lattice element $v \in \overset{\circ}{\sigma}$ has one of the following decomposition:

- $v = w + \sum_{i=1}^r \lambda_i b_i$ with $w \in \text{Box}(\sigma) \cap N$ and $\lambda_i \geq 0$ for all i ,
- $v = w + \sum_{i=1}^r \lambda_i b_i$ with $w \in \text{Box}^c(\sigma) \cap N - \{0\}$, $\lambda_i \geq 0$ for all $i \geq 2$ and $\lambda_1 > 0$,
- $v = \sum_{i=1}^r \lambda_i b_i$ where $\lambda_i > 0$ for all i

where $\text{Box}^c(\sigma)$ is the complement of $\text{Box}(\sigma)$ in $\square(\sigma)$. We get

$$(z-1)^r \sum_{v \in \overset{\circ}{\sigma} \cap N} z^{-\nu(v)} = \sum_{v \in \text{Box}(\sigma) \cap N} z^{r-\nu(v)} + \sum_{v \in \text{Box}^c(\sigma) \cap N - \{0\}} z^{r-1-\nu(v)} + 1 \quad (22)$$

because

- $\sum_{\lambda_1, \dots, \lambda_r \geq 0} z^{-\nu(w)} z^{-\lambda_1} \dots z^{-\lambda_r} = \frac{z^{r-\nu(w)}}{(z-1)^r}$ if $w \in \text{Box}(\sigma) \cap N$,
- $\sum_{\lambda_1 > 0, \lambda_2, \dots, \lambda_r \geq 0} z^{-\nu(w)} z^{-\lambda_1} \dots z^{-\lambda_r} = \frac{z^{r-1-\nu(w)}}{(z-1)^r}$ if $w \in \text{Box}^c(\sigma) \cap N - \{0\}$,
- $\sum_{\lambda_1, \dots, \lambda_r > 0} z^{-\lambda_1} \dots z^{-\lambda_r} = \frac{1}{(z-1)^r}$

(and we use the fact that $\nu(b_i) = 1$). Moreover,

- $\alpha \in \nu(\text{Box}(\sigma)) := \{\nu(v), v \in \text{Box}(\sigma)\}$ if and only if $r - \alpha \in \nu(\text{Box}(\sigma))$,
- $\alpha \in \nu(\text{Box}^c(\sigma)) := \{\nu(v), v \in \text{Box}^c(\sigma)\}$ if and only if $r - 1 - \alpha \in \nu(\text{Box}^c(\sigma))$.

because $q_i \in]0, 1[$ if and only if $1 - q_i \in]0, 1[$. We then deduce from (22) that

$$(z-1)^n \sum_{v \in \overset{\circ}{\sigma} \cap N} z^{-\nu(v)} = (z-1)^{n-r} \sum_{v \in \square(\sigma) \cap N} z^{\nu(v)} \quad (23)$$

for any $\sigma \in \Delta(r)$. The expected equality follows because the relative interiors of the cones of the complete fan Δ give a partition of its support. For the assertion about the Euler number, notice that

$$\lim_{z \rightarrow 1} \text{Spec}_P^{\text{geo}}(z) = \sum_{\sigma \in \Delta(n)} \sum_{v \in \square(\sigma) \cap N} 1 = n! \text{vol}(P)$$

because the normalized volume of $\sigma \cap \{v \in N_{\mathbb{R}}, \nu(v) \leq 1\}$ is equal to the number of lattice points in $\square(\sigma)$. \square

Proposition 4.2.2 *We have*

$$\text{Spec}_P^{\text{geo}}(z) = \sum_{\sigma \in \Delta} \sum_{v \in \text{Box}(\sigma) \cap N} h_\sigma(z) z^{\nu(v)}$$

where $h_\sigma(z) := \sum_{\tau \subseteq \sigma} (z-1)^{n-\dim \tau}$.

Proof. The expected equality follows from (21). \square

It turns out that $h_\sigma(z)$ is the Hodge-Deligne polynomial of the orbit closure $V(\sigma)$ (as defined for instance in [7, page 121]) of the orbit $O(\sigma)$. Because $V(\sigma)$ is a toric variety, the coefficients of $h_\sigma(z)$ are non-negative natural numbers (see for instance [25, Lemma 2.4] and the references therein) and we get $\text{Spec}_P^{\text{geo}}(z) = \sum_{i=1}^{\mu_P} z^{\beta_i}$ for a sequence $\beta_1, \dots, \beta_{\mu_P}$ of nonnegative rational numbers. We will also call this sequence the *geometric spectrum* of P .

Corollary 4.2.3 *The geometric spectrum satisfies $z^n \text{Spec}_P^{\text{geo}}(z^{-1}) = \text{Spec}_P^{\text{geo}}(z)$.*

Proof. Because $z^{n-\dim \sigma} h_\sigma(z^{-1}) = h_\sigma(z)$ (see again [25, Lemma 2.4]) and $\sum_i (1 - m_i) b_i \in \text{Box}(\sigma)$ if $\sum_i m_i b_i \in \text{Box}(\sigma)$ the assertion follows from proposition 4.2.2. \square

Corollary 4.2.4 *Let P be a full dimensional simplicial lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior, and $\Delta = (N, \Delta_P, \{b_i\})$ be its stacky fan. Then*

$$\text{Spec}_P^{\text{geo}}(z) = \sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{C}} H_{\text{orb}}^{2\alpha}(\mathcal{X}(\Delta), \mathbb{C}) z^\alpha$$

In words, the geometric spectrum of P is the Hilbert-Poincaré series of the graded vector space $H_{\text{orb}}^{2}(\mathcal{X}(\Delta), \mathbb{C})$.*

Proof. We have $H_{\text{orb}}^{2i}(\mathcal{X}(\Delta), \mathbb{Q}) = \oplus_{\sigma \in \Delta} \oplus_{v \in \text{Box}(\sigma) \cap N} H^{2(i-\nu(v))}(X_{\Delta/\sigma}, \mathbb{Q})$, see formula (16). It follows that the orbifold degrees are $\alpha = j + \nu(v)$ where $v \in \text{Box}(\sigma)$ and $j = 0, \dots, n - \dim \sigma$ and we get

$$\sum_{\alpha} \dim_{\mathbb{Q}} H_{\text{orb}}^{2\alpha}(\mathcal{X}(\Delta), \mathbb{Q}) z^\alpha = \sum_{\sigma \in \Delta} \sum_{v \in \text{Box}(\sigma) \cap N} \sum_{j=0}^{n-\dim \sigma} \dim_{\mathbb{Q}} H^{2j}(X_{\Delta/\sigma}, \mathbb{Q}) z^j z^{\nu(v)}$$

Now, $\sum_{j=0}^{n-\dim \sigma} \dim_{\mathbb{Q}} H^{2j}(X_{\Delta/\sigma}, \mathbb{Q}) z^j = h_\sigma(z)$ because the orbit closure $V(\sigma)$ and the toric variety $X_{\Delta/\sigma}$ are isomorphic (see for instance [7, Proposition 3.2.7]) and we get

$$\sum_{\alpha} \dim_{\mathbb{Q}} H_{\text{orb}}^{2\alpha}(\mathcal{X}(\Delta), \mathbb{Q}) z^\alpha = \sum_{\sigma \in \Delta} \sum_{v \in \text{Box}(\sigma) \cap N} h_\sigma(z) z^{\nu(v)}$$

The assertion then follows from proposition 4.2.2. \square

It follows that if P is smooth we have $\text{Spec}_P^{\text{geo}}(z) = \sum_{i=0}^n \dim_{\mathbb{C}} H^{2i}(X_{\Delta_P}, \mathbb{C}) z^i$: the geometric spectrum is thus the Poincaré polynomial in this case.

To sum up, the geometric spectrum of a simplicial polytope is a spectrum in the sense of section 3. Rationality, positivity and the volume property are given by lemma 4.2.1 and proposition 4.2.2, symmetry (Poincaré duality) by corollary 4.2.3 and modality by corollary 4.2.4.

4.2.2 Second interpretation: stacky E -function of a polytope (resolution of singularities)

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior. Let $\rho : Y \rightarrow X$ be a resolution of $X := X_{\Delta_P}$ as in section 2.5, ρ_1, \dots, ρ_r be the rays of Y with primitive generators v_1, \dots, v_r and associated divisors D_1, \dots, D_r and I . Put, for a subset $J \subset I := \{1, \dots, r\}$, $D_J := \cap_{j \in J} D_j$ if $J \neq \emptyset$ and $D_J := Y$ if $J = \emptyset$ and define

$$E_{st,P}(z) := \sum_{J \subset I} E(D_J, z) \prod_{j \in J} \frac{z - z^{\nu_j}}{z^{\nu_j} - 1} \quad (24)$$

where $\nu_j = \nu(v_j)$ and ν is the Newton function of P of definition 2.1.1.

Proposition 4.2.5 *We have $\text{Spec}_P^{geo}(z) = E_{st,P}(z)$. In particular, $E_{st,P}(z)$ does not depend on the resolution ρ .*

Proof. Using the notations of section 2.5 we have $E(D_J^\circ, z) = \sum_{J' \subset J} (-1)^{|J| - |J'|} E(D_{J'}, z)$ and

$$E_{st,P}(z) = \sum_{J \subset I} E(D_J^\circ, z) \prod_{j \in J} \frac{z - 1}{z^{\nu_j} - 1}$$

as in [1, Proof of theorem 3.7]. Let σ be a smooth cone of Δ' , the fan of Y , generated by v_{i_1}, \dots, v_{i_r} and $v \in \overset{\circ}{\sigma}$: we have $v = a_1 v_{i_1} + \dots + a_r v_{i_r}$ for $a_1, \dots, a_r > 0$ and $\nu(v) = a_1 \nu(v_{i_1}) + \dots + a_r \nu(v_{i_r})$. Thus

$$\sum_{v \in \overset{\circ}{\sigma} \cap N} z^{-\nu(v)} = \frac{1}{z^{\nu(v_{i_1})} - 1} \cdots \frac{1}{z^{\nu(v_{i_r})} - 1}$$

With these two observations in mind, the proof of the proposition is similar to the one of [1, Theorem 4.3]. \square

Remark 4.2.6 *Applying Poincaré duality to the smooth subvarieties D_J , we get once again the symmetry relation $z^n \text{Spec}_P^{geo}(z^{-1}) = \text{Spec}_P^{geo}(z)$ of corollary 4.2.3.*

4.2.3 Third interpretation: twisted δ -vector

Let P be a full dimensional lattice polytope in $N_{\mathbb{R}}$, containing the origin in its interior. Following [25], we define

$$F_P^0(z) = \sum_{m \geq 0} \sum_{v \in mP \cap N} z^{\nu(v) - \lceil \nu(v) \rceil + m}$$

which is a twisted version of the Ehrhart series $F_P(z)$ defined in section 2.2.

Proposition 4.2.7 *We have $\text{Spec}_P^{geo}(z) = (1 - z)^{n+1} F_P^0(z)$.*

Proof. Notice first that $v \in mP$ if and only if $\nu(v) \leq m$: this follows from the presentation (9) and the definition of the Newton function ν . We thus have

$$F_P^0(z^{-1}) = \sum_{m \geq 0} \sum_{\nu(v) \leq m} z^{-\nu(v) + [\nu(v)] - m} = \sum_{v \in N} \sum_{[\nu(v)] \leq m} z^{-\nu(v) + [\nu(v)] - m} = \frac{1}{1 - z^{-1}} \sum_{v \in N} z^{-\nu(v)}$$

and this gives $(z - 1)^n (1 - z^{-1}) F_P^0(z^{-1}) = \text{Spec}_P^{\text{geo}}(z)$. Thus

$$(1 - z)^{n+1} F_P^0(z) = z^n \text{Spec}_P^{\text{geo}}(z^{-1}) = \text{Spec}_P^{\text{geo}}(z)$$

where the last equality follows from corollary 4.2.3. \square

Corollary 4.2.8 *The part of the geometric spectrum contained in $[0, 1[$ is the sequence $\nu(v)$, $v \in \text{Int } P \cap N$. In particular, the multiplicity of 0 in the geometric spectrum is equal to one. Moreover the multiplicity of 1 in $\text{Spec}_P^{\text{geo}}$ is equal to $\text{Card}(\partial P \cap N) - n$.*

Proof. Scrutinization of the coefficients of z^a , $a \leq 1$, in the formula of proposition 4.2.7 (see also [25, Lemma 3.13]). \square

If P is reflexive, we have the following link between the δ -vector of P from section 2.2 and its geometric spectrum, see also [21]:

Corollary 4.2.9 *Let P be a reflexive full dimensional lattice polytope containing the origin in its interior. Then $\text{Spec}_P^{\text{geo}}(z) = \delta_0 + \delta_1 z + \dots + \delta_n z^n$ where $\delta = (\delta_0, \dots, \delta_n)$ is the δ -vector of P .*

Proof. By (10) we have $\nu(v) \in \mathbb{N}$ for all $v \in N$ because P is reflexive. We thus get $F_P^0(z) = F_P(z)$ where $F_P(z)$ is the Ehrhart series of P of section 2.2 because

$$F_P(z) = \sum_{m \geq 0} \text{Card}(mP \cap N) z^m = \sum_{m \geq 0} \sum_{v \in mP \cap N} z^m$$

By proposition 4.2.7 we have $\text{Spec}_P^{\text{geo}}(z) = (1 - z)^{n+1} F_P(z)$ and we use formula (12). \square

5 Algebraic spectrum of a polytope

Singularity theory associates to a (tame) Laurent polynomial function f a *spectrum at infinity*, see [24]. We recall its definition and its main properties in section 5.3. We can shift this notion to the Newton polytope P of f (P is assumed to be simplicial) and get in this way the *algebraic spectrum* of P . In order to motivate the next sections, we describe the Givental-Hori-Vafa models [15], [18] which are the expected mirror partners of toric varieties. In order to make the text as self-contained as possible, we first recall Kouchnirenko's results.

5.1 Preliminaries: Kouchnirenko's framework

We briefly recall the setting of [19]. Let $f : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ be a Laurent polynomial, $f(\underline{u}) = \sum_{a \in \mathbb{Z}^n} c_a \underline{u}^a$ where $\underline{u}^a := u_1^{a_1} \cdots u_n^{a_n}$. The Newton polytope P of f is the convex hull of the multi-indices a such that $c_a \neq 0$. We say that f is *convenient* if P contains the origin in its interior, *nondegenerate* if, for any face F of P , the system

$$u_1 \frac{\partial f_F}{\partial u_1} = \cdots = u_n \frac{\partial f_F}{\partial u_n} = 0$$

has no solution on $(\mathbb{C}^*)^n$ where $f_F(\underline{u}) = \sum_{a \in F \cap P} c_a \underline{u}^a$ the sum being taken over the multi-indices a such that $c_a \neq 0$. A convenient and nondegenerate Laurent polynomial f has only isolated critical points and its global Milnor number μ_f (the number of critical points with multiplicities) is $\mu_P := n! \text{vol}(P)$. Moreover, f is tame in the sense that the set outside which f is a locally trivial fibration is made from critical values of f , and these critical values belong to this set only because of the critical points at finite distance.

5.2 Givental-Hori-Vafa models and mirror symmetry

Let $N = \mathbb{Z}^n$, M be the dual lattice, Δ be a complete and simplicial fan and v_1, \dots, v_r be the primitive generators of its rays. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}^{r-n} \xrightarrow{\psi} \mathbb{Z}^r \xrightarrow{\varphi} \mathbb{Z}^n \longrightarrow 0$$

where $\varphi(e_i) = v_i$ for $i = 1, \dots, r$ ((e_i) denotes the canonical basis of \mathbb{Z}^r) and ψ describes the relations between the v_i 's. Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to this exact sequence, we get

$$1 \longrightarrow (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^r \xrightarrow{\pi} (\mathbb{C}^*)^{r-n} \longrightarrow 1$$

where

$$\pi(u_1, \dots, u_r) = (q_1, \dots, q_{r-n}) = (u_1^{a_{1,1}} \cdots u_r^{a_{r,1}}, \dots, u_1^{a_{1,r-n}} \cdots u_r^{a_{r,r-n}}) \quad (25)$$

and the integers $a_{i,j}$ satisfy $\sum_{j=1}^r a_{j,i} v_j = 0$ for $i = 1, \dots, r-n$. The *Givental-Hori-Vafa model* of the toric variety X_Δ is the function

$$u_1 + \cdots + u_r \text{ restricted to } U := \pi^{-1}(q_1, \dots, q_{r-n})$$

We will denote it by f_{X_Δ} . The stacky version of this construction is straightforward: replace the v_i 's by the b_i 's.

If Δ contains a smooth cone, the function f_{X_Δ} is easily described:

Proposition 5.2.1 *Assume that (v_1, \dots, v_n) is the canonical basis of N . Then f_{X_Δ} is the Laurent polynomial defined on $(\mathbb{C}^*)^n$ by*

$$f_{X_\Delta}(u_1, \dots, u_n) = u_1 + \cdots + u_n + \sum_{i=n+1}^r q_i u_1^{v_1^i} \cdots u_n^{v_n^i}$$

if $v_i = (v_1^i, \dots, v_n^i) \in \mathbb{Z}^n$ for $i = n+1, \dots, r$.

Proof. We have $v_i = \sum_{j=1}^n v_j^i v_j$ for $i = n+1, \dots, r$ and we use presentation (25). \square

Above a convenient and non degenerate Laurent polynomial f we make grow a differential system (see [12] and also [11] for explicit computations on weighted projective spaces) and we say that f is a mirror partner of a variety X if this differential system is isomorphic to the one associated with the (small quantum, orbifold) cohomology of X . If f is the mirror partner of a smooth variety X the following properties are in particular expected (non-exhaustive list):

- the Milnor number of f is equal to the rank of the cohomology of X ,
- the spectrum at infinity of f (see section 5.3 below) is equal to half of the degrees of the cohomology groups of X ,
- multiplication by f on its Jacobi ring yields the cup-product by $c_1(X)$ on the cohomology algebra of X .

The first thing to do is to compare the dimension of the Jacobi ring of f_{X_Δ} , hence its Milnor number $\mu_{f_{X_\Delta}}$, and the rank of the cohomology algebra of X_Δ . In the smooth case, we have equality if (and only if) X_Δ is weak Fano. If X_Δ is smooth, complete, but not weak Fano we have $\mu_{f_{X_\Delta}} > \chi(X_\Delta)$: see section 5.5 for a picture of this phenomenon.

In the singular simplicial case (*i.e* X is an orbifold), cohomology should be replaced by orbifold cohomology, see section 6: the rank of the orbifold cohomology is not a number of cones but a normalized volume, and the cohomology degrees are the orbifold degrees.

5.3 The spectrum at infinity of a tame Laurent polynomial

We assume in this section that f is a convenient and nondegenerate Laurent polynomial, defined on $U := (\mathbb{C}^*)^n$, with global Milnor number μ . For the (very small) D -module part, we use the notations of [12, 2.c]. Let G be the Fourier-Laplace transform of the Gauss-Manin system of f , G_0 be its Brieskorn lattice (G_0 is indeed a free $\mathbb{C}[\theta]$ -module because f is convenient and nondegenerate and $G = \mathbb{C}[\theta, \theta^{-1}] \otimes G_0$, see [12, Remark 4.8]) and V_\bullet be the V -filtration of G at infinity, that is along $\theta^{-1} = 0$. From these data we get by projection a V -filtration on the μ -dimensional vector space $\Omega_f := \Omega^n(U)/df \wedge \Omega^{n-1}(U) = G_0/\theta G_0$, see [12, Section 2.e].

The *spectrum at infinity* of f is the spectrum of the V -filtration defined on Ω_f , that is the (ordered) sequence $\alpha_1, \alpha_2, \dots, \alpha_\mu$ of rational numbers with the following property: the frequency of α in the sequence is equal to $\dim_{\mathbb{C}} gr_V^\alpha \Omega_f$. We will denote it by $Spec_f$ and we will write $Spec_f(z) = \sum_{i=1}^\mu z^{\alpha_i}$. Recall the following facts, see [24]: we have $\alpha_i \geq 0$ for all i and $Spec_f(z) = z^n Spec_f(z^{-1})$. In particular, $Spec_f \subset [0, n]$.

If f is convenient and nondegenerate, its spectrum at infinity can be computed using the Newton function of its Newton polytope: let us define the Newton filtration \mathcal{N}_\bullet on $\Omega^n(U)$ by

$$\mathcal{N}_\alpha \Omega^n(U) = \left\{ \sum_{c \in N} a_c \omega_c, \nu(\omega_c) \leq \alpha \text{ for all } c \text{ such that } a_c \neq 0 \right\}$$

where $\nu(\omega_c) := \nu(c)$ if $\omega_c = u_1^{c_1} \dots u_n^{c_n} \frac{du_1 \wedge \dots \wedge du_n}{u_1 \dots u_n}$ and $c = (c_1, \dots, c_n) \in N$ (notice the normalization $\nu(\frac{du_1 \wedge \dots \wedge du_n}{u_1 \dots u_n}) = 0$). This filtration induces a filtration on Ω_f by projection and the spectrum at infinity of f is equal to the spectrum of this filtration, see [12, Corollary 4.13].

5.4 The algebraic spectrum of a polytope

We define the algebraic spectrum $\text{Spec}_P^{\text{alg}}$ of a simplicial full dimensional lattice polytope P containing the origin in its interior to be the spectrum at infinity of the Laurent polynomial $f_P(\underline{u}) = \sum_{b \in \mathcal{V}(P)} \underline{u}^b$ where $\mathcal{V}(P)$ denotes the set of the vertices of P . Notice that f_P is a convenient and nondegenerate Laurent polynomial and that its Milnor number is $\mu_{f_P} = \mu_P$: indeed, f_P is convenient by definition because P contains the origin in its interior and it is nondegenerate because of the simpliciality assumption; the assertion about the Milnor number then follows from [19]. We identify the algebraic spectrum with its generating function $\text{Spec}_P^{\text{alg}}(z) = \sum_{i=1}^{\mu} z^{\alpha_i}$. We have $\text{Spec}_P^{\text{alg}}(z) = z^n \text{Spec}_P^{\text{alg}}(z^{-1})$.

Proposition 5.4.1 *Let P be a full dimensional simplicial lattice polytope in $N_{\mathbb{R}}$ containing the origin in its interior. Then the part of the algebraic spectrum contained in $[0, 1[$ is the sequence $\nu(v)$, $v \in \text{Int } P \cap N$ where ν is the Newton function of P of definition 2.1.1. In particular, the multiplicity of 0 in $\text{Spec}_P^{\text{alg}}$ is equal to one.*

Proof. The assertion for Spec_{f_P} follows from [12, Lemma 4.6], as in [12, Example 4.17]. \square

In the two dimensional case, we deduce the following description of the algebraic spectrum:

Proposition 5.4.2 *Let P be a full dimensional lattice polytope in $N_{\mathbb{R}} = \mathbb{R}^2$ containing the origin in its interior. Then*

$$\text{Spec}_P^{\text{alg}}(z) = (\text{Card}(\partial P \cap N) - 2)z + \sum_{v \in \text{Int } P \cap N} (z^{\nu(v)} + z^{2-\nu(v)})$$

where ν is the Newton function of P .

Proof. Let $f_P(\underline{u}) = \sum_{b \in \mathcal{V}(P)} \underline{u}^b$ be as above. From proposition 5.4.1, and because $\text{Spec}_{f_P}(z) = z^2 \text{Spec}_{f_P}(z^{-1})$, we get

$$\text{Spec}_{f_P}(z) = (\text{Card}(\partial P \cap N) - 2)z + \sum_{v \in \text{Int } P \cap N} (z^{\nu(v)} + z^{2-\nu(v)})$$

where the coefficient of z is computed using Pick's formula because $\mu_{f_P} = 2 \text{vol}(P)$ by [19]. \square

In any dimension, we also have the following description for reflexive polytopes:

Proposition 5.4.3 *Let P be a full dimensional reflexive simplicial polytope in $N_{\mathbb{R}} = \mathbb{R}^n$ containing the origin in its interior. Then:*

- $\text{Spec}_P^{\text{alg}}(z) = \sum_{i=0}^n d(i)z^i$ where $d(i) \in \mathbb{N}$,
- $d(i) = d(n - i)$ for $i = 0, \dots, n$ with $d(0) = d(n) = 1$,
- $\sum_{i=0}^n d(i) = \mu_P$

Proof. Because P is reflexive, the Newton function takes integer values at the lattice points, see (10). This gives the first point because $\text{Spec}_P^{\text{alg}} \subset [0, n]$. For the second one, use the symmetry and the fact that 0 is in the spectrum with multiplicity one. \square

5.5 Examples: Hirzebruch surfaces and their Givental-Hori-Vafa models

Let m be a positive integer. The fan $\Delta_{\mathbb{F}_m}$ of the Hirzebruch surface \mathbb{F}_m is the one whose rays are generated by the vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, m)$, $v_4 = (0, -1)$, see for instance [14]. The surface \mathbb{F}_m is Fano if $m = 1$, weak Fano if $m = 2$. Its Givental-Hori-Vafa model is the Laurent polynomial

$$f_m(u_1, u_2) = u_1 + u_2 + \frac{q_1}{u_2} + q_2 \frac{u_2^m}{u_1}$$

defined on $(\mathbb{C}^*)^2$, where q_1 and q_2 are two non zero parameters. We have

1. $\mu_{f_1} = 4$ and $\text{Spec}_{f_1}(z) = 1 + 2z + z^2$,
2. $\mu_{f_2} = 4$ if $q_2 \neq \frac{1}{4}$ and $\text{Spec}_{f_2}(z) = 1 + 2z + z^2$,
3. $\mu_{f_m} = m + 2$ if $m \geq 3$ and

$$\text{Spec}_{f_m}(z) = 1 + 2z + z^2 + z^{\frac{1}{p}} + z^{\frac{2}{p}} + \dots + z^{\frac{p-1}{p}} + z^{2-\frac{1}{p}} + z^{2-\frac{2}{p}} + \dots + z^{2-\frac{p-1}{p}}$$

if $m = 2p$ and $p \geq 2$,

$$\text{Spec}_{f_m}(z) = 1 + z + z^2 + z^{\frac{2}{m}} + z^{\frac{4}{m}} + \dots + z^{\frac{2p}{m}} + z^{2-\frac{2}{m}} + z^{2-\frac{4}{m}} + \dots + z^{2-\frac{2p}{m}}$$

if $m = 2p + 1$ and $p \geq 1$.

Indeed, for $m \neq 2$ we have $\mu_{f_m} = 2! \text{vol}(P)$, where P is the Newton polytope of f_m , because f_m is convenient and nondegenerate for all non zero value of the parameters, see section 5.1. For $m = 2$, f_2 is nondegenerate if and only if $q_2 \neq \frac{1}{4}$ and the previous argument applies in this case (if $q_2 = 1/4$ the Milnor number is 2). The spectrum is given by proposition 5.4.2.

The function f_2 is a guenuine mirror partner of the surface \mathbb{F}_2 , see [9], [23]. If $m \geq 3$, we have $\mu_{f_m} > 4$ and the model f_m has too many critical points: because \mathbb{F}_m is not weak Fano in this case, this is consistent with the results of section 5.2.

6 Geometric spectrum vs algebraic spectrum

We show in this section the equality $\text{Spec}_P^{\text{alg}}(z) = \text{Spec}_P^{\text{geo}}(z)$, see corollary 6.1.2 (in the two dimensional case, this follows immediately from corollary 4.2.8, proposition 5.4.1, and the symmetry property). The idea is to show that these functions are Hilbert-Poincaré series of isomorphic graded rings (theorem 6.1.1). This is achieved using the description of the orbifold Chow ring given in [4].

6.1 An isomorphism of graded rings

Let P be a simplicial polytope containing the origin in its interior, $f_P(\underline{u}) = \sum_{j=1}^r \underline{u}^{b_j}$ be the corresponding convenient and nondegenerate Laurent polynomial as in section 5.4. In what follows, we will write $\underline{u}^c := u_1^{c_1} \dots u_n^{c_n}$ if $c = (c_1, \dots, c_n) \in N$ and $K := \mathbb{Q}[u_1, u_1^{-1}, \dots, u_n, u_n^{-1}]$. Let

$$\mathcal{A}_{f_P} = \frac{K}{\langle u_1 \frac{\partial f_P}{\partial u_1}, \dots, u_n \frac{\partial f_P}{\partial u_n} \rangle}$$

be the jacobian ring of f_P . Define on K the Newton filtration \mathcal{N}_\bullet by

$$\mathcal{N}_\alpha K = \left\{ \sum_{c \in N} a_c \underline{u}^c \in K, \nu(c) \leq \alpha \text{ for all } c \text{ such that } a_c \neq 0 \right\}$$

where ν is the Newton function of P (the vector space $\mathcal{N}_{<\alpha} K$ is defined similarly: replace the condition $\nu(c) \leq \alpha$ by $\nu(c) < \alpha$). This filtration induces by projection a filtration, also denoted by \mathcal{N}_\bullet on \mathcal{A}_{f_P} , and we get the graded ring $Gr_{\mathcal{N}} \mathcal{A}_{f_P} = \bigoplus_{\alpha} \mathcal{N}_{\alpha} \mathcal{A}_{f_P} / \mathcal{N}_{<\alpha} \mathcal{A}_{f_P}$ (see equation (27) below).

Theorem 6.1.1 *There is an isomorphism of graded rings*

$$A_{orb}^*(\mathcal{X}(\Delta)) \cong Gr_{\mathcal{N}} \mathcal{A}_{f_P}$$

where Δ is the stacky fan associated with P (see section 2.4).

Proof. We first recall the setting of [4, section 5]. Let $\Delta = (N, \Delta, \{b_i\})$ be the stacky fan of P . We define its deformed group ring $\mathbb{Q}[N]^\Delta$ as follows:

- as a \mathbb{Q} -vector space, $\mathbb{Q}[N]^\Delta = \bigoplus_{c \in N} \mathbb{Q} y^c$ where y is a formal variable,
- the ring structure is given by $y^{c_1} \cdot y^{c_2} = y^{c_1+c_2}$ if c_1 and c_2 belong to a same cone, $y^{c_1} \cdot y^{c_2} = 0$ otherwise,
- the grading is defined as follows: if $c = \sum_{\rho_i \subseteq \sigma(c)} m_i b_i$ then $\deg(y^c) = \sum m_i \in \mathbb{Q}$ ($\sigma(c)$ is the minimal cone containing c).

It will be important to notice that $\deg(y^c) = \nu(c)$ where ν is the Newton function. Because Δ is simplicial, we have an isomorphism of \mathbb{Q} -graded rings

$$\frac{\mathbb{Q}[N]^\Delta}{\langle \sum_{i=1}^r \langle m, b_i \rangle y^{b_i}, m \in M \rangle} \longrightarrow \bigoplus_{v \in \square(\Delta)} A^*(\mathcal{X}(\Delta/\sigma(v)))[\deg(y^v)] \quad (26)$$

where $\sigma(v)$ is the minimal cone containing v , $\square(\sigma) := \{\sum_{\rho_i \subseteq \sigma} \lambda_i b_i, \lambda_i \in [0, 1]\}$ and $\square(\Delta)$ is the union of $\square(\sigma)$ for all n -dimensional cones $\sigma \in \Delta$, see [4, Theorem 1.1].

On the other side, we have

$$\mathcal{A}_{f_P} = \frac{K}{\langle \sum_{j=1}^r \langle m, b_j \rangle \underline{u}^{b_j}, m \in M \rangle}$$

because $u_i \frac{\partial f_P}{\partial u_i} = \sum_{j=1}^r \langle e_i^*, b_j \rangle \underline{u}^{b_j}$ where (e_i^*) is the dual basis of the canonical basis of N . Define the ring

$$A_{f_P} = \frac{K^g}{\langle \sum_{j=1}^r \langle m, b_j \rangle \underline{u}^{b_j}, m \in M \rangle}$$

where $K^g = K$ as a vector space and the multiplication on K^g is defined as follows:

$$\underline{u}^{c_1} \cdot \underline{u}^{c_2} = \begin{cases} \underline{u}^{c_1+c_2} & \text{if } c_1 \text{ and } c_2 \text{ belong to a same cone } \sigma \text{ of } \Delta_P, \\ 0 & \text{otherwise} \end{cases}$$

Define a grading on K^g by $\deg(\underline{u}^c) = \nu(c)$. Because $\nu(c_1 + c_2) = \nu(c_1) + \nu(c_2)$ if and only if c_1 and c_2 belong to a same cone σ of Δ_P and because $\nu(b_j) = 1$, the ring A_{f_P} is graded. Moreover, because

$$\underline{u}^v \cdot \underline{u}^w \in \begin{cases} \mathcal{N}_{\alpha+\beta}K & \text{if } \underline{u}^v \in \mathcal{N}_\alpha K \text{ and } \underline{u}^w \in \mathcal{N}_\beta K, \\ \mathcal{N}_{<\alpha+\beta}K & \text{if } v \text{ and } w \text{ do not belong to a same cone of } \Delta_P \end{cases} \quad (27)$$

the graded ring A_{f_P} is isomorphic to $Gr_{\mathcal{N}}\mathcal{A}_{f_P}$. Last, the rings $\frac{\mathbb{Q}[N]^\Delta}{\langle \sum_{i=1}^r \langle m, b_i \rangle y^{b_i}, m \in M \rangle}$ and A_{f_P} are isomorphic, and the theorem follows now from the isomorphism (26). \square

Corollary 6.1.2 *Assume that P is a simplicial polytope containing the origin in its interior. Then $Spec_P^{alg}(z) = Spec_P^{geo}(z)$.*

Proof. As explained in section 5.3, $Spec_P^{alg}(z)$ is the Hilbert-Poincaré series of the graded vector space $Gr_{\mathcal{N}}\mathcal{A}_{f_P}$. Now, theorem 6.1.1 shows that the latter coincide with the Hilbert-Poincaré series of $A_{orb}^*(\mathcal{X}(\Delta))$ and we get the assertion by corollary 4.2.4. \square

6.2 A significant class of examples: weighted projective spaces

Let $(\lambda_0, \dots, \lambda_n) \in (\mathbb{N}^*)^{n+1}$ such that $\gcd(\lambda_0, \dots, \lambda_n) = 1$ and X be the weighted projective space $\mathbb{P}(\lambda_0, \dots, \lambda_n)$. The (stacky) fan of X is the simplicial complete fan whose rays are generated by vectors b_0, \dots, b_n in N such that

1. $\lambda_0 b_0 + \dots + \lambda_n b_n = 0$
2. the b_i 's generate N

Such a family is unique, up to isomorphism. We have $\lambda_0 = 1$ if and only if (b_1, \dots, b_n) is a basis of N and this will be our favorite situation: *we assume from now on that $\lambda_0 = 1$* . In this situation, we will call the convex hull P of b_0, \dots, b_n *the polytope of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$* . We have $\mu_P = 1 + \lambda_1 + \dots + \lambda_n$ and $\mu_{P^\circ} = \frac{(1 + \lambda_1 + \dots + \lambda_n)^n}{\lambda_1 \dots \lambda_n}$. The polytope P is reflexive if and only if λ_i divides μ_P for all i .

By proposition 5.2.1, the Givental-Hori-Vafa model of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$ is the Laurent polynomial defined on $(\mathbb{C}^*)^n$ by

$$f(u_1, \dots, u_n) = u_1 + \dots + u_n + \frac{q}{u_1^{\lambda_1} \dots u_n^{\lambda_n}}$$

where $q \in \mathbb{C}^*$. Its Milnor number is $\mu_f = 1 + \lambda_1 + \dots + \lambda_n = \mu_P$. A mirror theorem is shown in [11].

Let

$$F := \left\{ \frac{\ell}{\lambda_i} \mid 0 \leq \ell \leq \lambda_i - 1, 0 \leq i \leq n \right\}.$$

and f_1, \dots, f_k the elements of F arranged by increasing order. We then define

$$S_{f_i} := \{j \mid \lambda_j f_i \in \mathbb{Z}\} \subset \{0, \dots, n\} \text{ and } d_i := \text{Card } S_{f_i}$$

Let $c_0, c_1, \dots, c_{\mu-1}$ be the sequence

$$\underbrace{f_1, \dots, f_1}_{d_1}, \underbrace{f_2, \dots, f_2}_{d_2}, \dots, \underbrace{f_k, \dots, f_k}_{d_k}$$

arranged by increasing order. By [13, Theorem 1], the spectrum at infinity of f is the sequence $\alpha_0, \alpha_1, \dots, \alpha_{\mu-1}$ where

$$\alpha_k := k - \mu c_k \text{ for } k = 0, \dots, \mu - 1$$

Notice that the spectrum of f is integral if and only if the polytope of $\mathbb{P}(1, \lambda_1, \dots, \lambda_n)$ is reflexive.

Example 6.2.1 *We test corollary 6.1.2 on some weighted projective spaces. The computation of the geometric spectrum is done using proposition 4.2.5.*

1. *Let a be a positive integer, and P be the convex hull of $(1, 0)$, $(-1, -a)$ and $(0, 1)$: this is the polytope of $\mathbb{P}(1, 1, a)$. We consider the resolution obtained by adding the ray generated by $(0, -1)$. Using the notations of proposition 4.2.5, we have $\nu_1 = 1$, $\nu_2 = \frac{2}{a}$, $\nu_3 = 1$ and $\nu_4 = 1$ (the rays are numbered clockwise) and we get (\mathbb{F}_a is the Hirzebruch surface)*

$$\begin{aligned} \text{Spec}_P^{\text{geo}}(z) &= E(\mathbb{F}_a, z) + E(\mathbb{P}^1, z) \frac{z - z^{2/a}}{z^{2/a} - 1} = 1 + 2z + z^2 + (1 + z) \left(\frac{z - 1}{z^{2/a} - 1} - 1 \right) \\ &= 1 + z + z^2 + z^{2/a} + z^{4/a} + \dots + z^{2(a-1)/a} = \text{Spec}_P^{\text{alg}}(z) \end{aligned}$$

2. *Let P be the convex hull of $(1, 0)$, $(-2, -5)$ and $(0, 1)$: P is the polytope of $\mathbb{P}(1, 2, 5)$. We consider the resolution obtained by adding the ray generated by $(0, -1)$, $(-1, -3)$ and $(-1, -2)$. Using the notations of proposition 4.2.5, we have $\nu_1 = 1$, $\nu_2 = \frac{3}{5}$, $\nu_3 = \frac{4}{5}$, $\nu_4 = 1$, $\nu_5 = 1$ and $\nu_6 = 1$ (the rays are numbered clockwise) and we get*

$$\begin{aligned} \text{Spec}_P^{\text{geo}}(z) &= z^2 + 4z + 1 + (z + 1) \frac{z - z^{3/5}}{z^{3/5} - 1} + (z + 1) \frac{z - z^{4/5}}{z^{4/5} - 1} + \frac{z - z^{3/5}}{z^{3/5} - 1} \cdot \frac{z - z^{4/5}}{z^{4/5} - 1} \\ &= z^2 + 2z + 1 + z^{3/5} + z^{4/5} + z^{6/5} + z^{7/5} = \text{Spec}_P^{\text{alg}}(z) \end{aligned}$$

3. *Let $\ell \in \mathbb{N}^*$ and P be the convex hull of $(1, 0)$, $(-\ell, -\ell)$ and $(0, 1)$: P is the polytope of $\mathbb{P}(1, \ell, \ell)$. The variety X_{Δ_P} is \mathbb{P}^2 , generated by the rays $v_1 = (-1, -1)$, $v_2 = (0, 1)$ and $v_3 = (1, 0)$. Because $\nu(v_1) = \frac{1}{\ell}$, we get*

$$\begin{aligned} \text{Spec}_P^{\text{geo}}(z) &= E(\mathbb{P}^2, z) + E(\mathbb{P}^1, z) \frac{z - z^{1/\ell}}{z^{1/\ell} - 1} = 1 + z + z^2 + (1 + z) \frac{z - z^{1/\ell}}{z^{1/\ell} - 1} \\ &= z^2 + z + 1 + z^{1/\ell} + \dots + z^{(\ell-1)/\ell} + z^{1+1/\ell} + \dots + z^{1+(\ell-1)/\ell} = \text{Spec}_P^{\text{alg}}(z) \end{aligned}$$

4. *Let P be the convex hull of $(-2, -2, -2)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$: P is the polytope of $\mathbb{P}(1, 2, 2, 2)$. We have $X_{\Delta_P} = \mathbb{P}^3$, generated by the rays $v_1 = (-1, -1, -1)$, $v_2 = (1, 0, 0)$, $v_3 = (0, 1, 0)$ and $v_4 = (0, 0, 1)$. Because $\nu(v_1) = \frac{1}{2}$, we get*

$$\begin{aligned} \text{Spec}_P^{\text{geo}}(z) &= E(\mathbb{P}^3, z) - E(\mathbb{P}^2, z) + E(\mathbb{P}^2, z) \frac{z - 1}{z^{1/2} - 1} \\ &= z^3 + z^2 + z + 1 + z^{1/2} + z^{3/2} + z^{5/2} = \text{Spec}_P^{\text{alg}}(z) \end{aligned}$$

7 A formula for the variance of the spectra

We are now ready to prove formula (4) of the introduction. We first show it for the geometric spectrum of a full dimensional simplicial lattice polytope.

7.1 Libgober-Wood's formula for the spectra

In order to get first a stacky version of the Libgober-Wood formula (1), we give the following definition, inspired by Batyrev's stringy number $c_{st}^{1,n-1}(X)$, see [2, Definition 3.1]:

Definition 7.1.1 *Let P be a full dimensional simplicial lattice polytope in N containing the origin in its interior and let $\rho : Y \rightarrow X$ be a resolution of $X := X_{\Delta_P}$. We define the rational number*

$$\begin{aligned} \widehat{\mu}_P := & c_1(Y)c_{n-1}(Y) + \sum_{J \subset I, J \neq \emptyset} c_1(D_J)c_{n-|J|-1}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \\ & - \sum_{J \subset I, J \neq \emptyset} \left(\sum_{j \in J} \nu_j \right) c_{n-|J|}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \end{aligned} \quad (28)$$

where the notations in the right hand term are the ones of section 4.2.2 (convention: $c_r(D_J) = 0$ if $r < 0$).

Remark 7.1.2 *We have $\widehat{\mu}_P = c_1(Y)c_{n-1}(Y)$ if $\nu_i = 1$ for all i (crepant resolutions) and $\widehat{\mu}_P = c_1(X)c_{n-1}(X)$ if X is smooth.*

Theorem 7.1.3 *Let P be a full dimensional simplicial lattice polytope in N containing the origin in its interior and $\text{Spec}_P^{geo}(z) = \sum_{i=1}^{e_P} z^{\beta_i}$ be its geometric spectrum. Then*

$$\sum_{i=1}^{e_P} \left(\beta_i - \frac{n}{2} \right)^2 = \frac{n}{12} \mu_P + \frac{1}{6} \widehat{\mu}_P \quad (29)$$

where $\widehat{\mu}_P$ is defined by formula (28) and $\mu_P := n! \text{vol}(P)$.

Proof. Recall the stacky E -fonction $E_{st,P}(z) := \sum_{J \subset I} E(D_J, z) \prod_{j \in J} \frac{z - z^{\nu_j}}{z^{\nu_j} - 1}$, see (24). Then we have

$$E''_{st,P}(1) = \frac{n(3n-5)}{12} e_P + \frac{1}{6} \widehat{\mu}_P \quad (30)$$

where e_P is the geometric Euler number of P , see definition 4.1.1. The proof of this formula is a straightforward computation and is similar to the one of [2, Theorem 3.8]: if V is a smooth variety of dimension n , we have the Libgober-Wood formula

$$E''(V, 1) = \frac{n(3n-5)}{12} c_n(V) + \frac{1}{6} c_1(V)c_{n-1}(V) \quad (31)$$

where E is the E -polynomial of V , see [20, Proposition 2.3]; in order to get (30), apply this formula to the components $E(D_J, z)$ of $E_{st,P}(z)$ and use the equalities

$$E(D_J, 1) = c_{n-|J|}(D_J), \quad \frac{d}{dz}(E(D_J, z))|_{z=1} = \frac{n-|J|}{2} c_{n-|J|}(D_J)$$

(the first one follows from the fact that the value at $z = 1$ of the Poincaré polynomial is the Euler characteristic and we get the second one using Poincaré duality for D_J) and

$$\frac{d}{dz}\left(\frac{z - z^\nu}{z^\nu - 1}\right)|_{z=1} = \frac{1 - \nu}{2\nu}, \quad \frac{d^2}{dz^2}\left(\frac{z - z^\nu}{z^\nu - 1}\right)|_{z=1} = \frac{(\nu - 1)(\nu + 1)}{6\nu}$$

if ν a positive rational number. By proposition 4.2.5, we have $\text{Spec}_P^{\text{geo}}(z) = E_{st,P}(z)$. Finally, we get

$$\frac{d^2}{dz^2}(\text{Spec}_P^{\text{geo}}(z))|_{z=1} = \frac{n(3n - 5)}{12}e_P + \frac{1}{6}\widehat{\mu}_P \quad (32)$$

We have $\frac{d}{dz}(\text{Spec}_P^{\text{geo}}(z))|_{z=1} = \frac{n}{2}e_P$, because the geometric spectrum is symmetric with respect to $\frac{n}{2}$ (see corollary 4.2.3), and we deduce that

$$\frac{d^2}{dz^2}(\text{Spec}_P^{\text{geo}}(z))|_{z=1} = \sum_{i=1}^{e_P} (\beta_i - \frac{n}{2})^2 + \frac{n(n - 2)}{4}e_P \quad (33)$$

Now, formulas (32) and (33) give equality (29) because $e_P = \mu_P$ by lemma 4.2.1. \square

Corollary 7.1.4 *The number $\widehat{\mu}_P$ does not depend on the resolution ρ .* \square

The version for singularities is then given by the following result:

Theorem 7.1.5 *Let f be a convenient and nondegenerate Laurent polynomial with global Milnor number μ and spectrum at infinity $\text{Spec}_f(z) = \sum_{i=1}^{\mu} z^{\alpha_i}$. Let P be its Newton polytope (assumed to be simplicial). Then*

$$\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \frac{n}{12}\mu_P + \frac{1}{6}\widehat{\mu}_P \quad (34)$$

where $\widehat{\mu}_P$ is defined by formula (28) and $\mu_P := n! \text{vol}(P) = \mu$.

Proof. By [22] we have $\text{Spec}_P^{\text{alg}}(z) = \text{Spec}_f(z)$ and the assertion thus follows from theorem 7.1.3 and corollary 6.1.2. Last, because f is convenient and nondegenerate we have $\mu = \mu_P$ by [19]. \square

Example 7.1.6 *Assume that f is the mirror partner of a projective, smooth, weak Fano toric variety X of dimension n (mirror symmetry is discussed in section 5.2). Then*

$$\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \mu \frac{n}{12} + \frac{1}{6}c_1(X)c_{n-1}(X) \quad (35)$$

Indeed, the convex hull P of the primitive generators of the rays of the fan of X is reflexive because X is weak Fano and (35) follows from theorem 7.1.5 and remark 7.1.2. We have $c_1(X)c_{n-1}(X) \geq 0$ because X is weak Fano and we get in particular

$$\sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 \geq \mu \frac{n}{12}$$

This is a first step towards Hertling's conjecture, see section 9.

7.2 The number $\hat{\mu}_P$ in the two dimensional case

In this section we give an explicit formula for $\hat{\mu}_P$ in the two dimensional case. Let P be a full dimensional polytope in $N_{\mathbb{R}} = \mathbb{R}^2$, containing the origin, and $\rho : Y \rightarrow X$ be a resolution of $X := X_{\Delta_P}$ as in section 4.2.2. We assume that the primitive generators v_1, \dots, v_r of the rays of Y are numbered clockwise and we consider indices as integers modulo r so that $\nu_{r+1} := \nu_1$ (recall that $\nu_i := \nu(v_i)$ where ν is the Newton function of P).

Proposition 7.2.1 *We have*

$$\hat{\mu}_P = c_1^2(Y) - 2r + \sum_{i=1}^r \left(\frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) = \left(\sum_{i=1}^r \nu_i D_i \right) \left(\sum_{j=1}^r \frac{1}{\nu_j} D_j \right)$$

Proof. By definition, we have

$$\begin{aligned} \hat{\mu}_P := c_1(Y)c_1(Y) + \sum_{J \subset I, J \neq \emptyset} c_1(D_J)c_{1-|J|}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \\ - \sum_{J \subset I, J \neq \emptyset} \left(\sum_{j \in J} \nu_j \right) c_{2-|J|}(D_J) \prod_{j \in J} \frac{1-\nu_j}{\nu_j} \end{aligned} \quad (36)$$

and thus

$$\hat{\mu}_P = c_1^2(Y) + 2 \sum_{i=1}^r \frac{(1-\nu_i)^2}{\nu_i} - \sum_{i=1}^r (\nu_i + \nu_{i+1}) \frac{(1-\nu_i)}{\nu_i} \cdot \frac{(1-\nu_{i+1})}{\nu_{i+1}}$$

It follows that

$$\begin{aligned} \hat{\mu}_P - c_1^2(Y) &= \sum_{i=1}^r \left(\frac{1}{\nu_i} + \nu_i - \frac{1}{\nu_{i+1}} - \nu_{i+1} + \frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} - 2 \right) \\ &= -2r + \sum_{i=1}^r \left(\frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) \end{aligned}$$

and this gives the first equality. For the second one, notice that

$$\begin{aligned} \left(\sum_{i=1}^r \nu_i D_i \right) \left(\sum_{j=1}^r \frac{1}{\nu_j} D_j \right) &= \sum_{i=1}^r \left(D_i^2 + \frac{\nu_i}{\nu_{i+1}} D_i D_{i+1} + \frac{\nu_i}{\nu_{i-1}} D_i D_{i-1} \right) \\ &= \sum_{i=1}^r \left(D_i^2 + \frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) = c_1^2(Y) - 2r + \sum_{i=1}^r \left(\frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i} \right) = \hat{\mu}_P \end{aligned}$$

□

Corollary 7.2.2 *Let P be a full dimensional lattice polytope in $N_{\mathbb{R}} = \mathbb{R}^2$ containing the origin in its interior and $\text{Spec}_P^{\text{geo}}(z) = \sum_{i=1}^{e_P} z^{\beta_i}$ be its geometric spectrum. Then*

$$\sum_{i=1}^{\mu_P} (\beta_i - 1)^2 = \frac{\mu_P}{6} + \frac{\hat{\mu}_P}{6} \quad (37)$$

where $\hat{\mu}_P \geq c_1^2(Y)$ for any resolution $\rho : Y \rightarrow X_{\Delta_P}$.

Proof. The first equality follows from theorem 7.1.3. By proposition 7.2.1 we have $\widehat{\mu}_P \geq c_1^2(Y)$ because $\nu + \frac{1}{\nu} \geq 2$ for all real positive number ν . \square

In particular, $\sum_{i=1}^{\mu} (\alpha_i - 1)^2 \geq \frac{\mu_P}{6}$ if there exists a resolution ρ such that $c_1^2(Y) \geq 0$. The singularity version is straightforward.

7.3 Examples

7.3.1 Weighted projective spaces (examples 6.2.1 continued)

We test theorem 7.1.5 on weighted projective spaces: given $X = \mathbb{P}(1, \lambda_1, \dots, \lambda_n)$, P will denote its polytope, $f(\underline{u}) = \sum_{i=0}^n \underline{u}^{b_i}$ will denote its Givental-Hori-Vafa model with spectrum at infinity $\sum_{i=1}^{\mu} z^{\alpha_i}$ as in section 6.2. We put $V(\alpha) := \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2$.

- The polytope P is Fano (see section 2.3):

X	μ	$V(\alpha)$	$\mu n/12$	$\widehat{\mu}_P$
$\mathbb{P}(1, 1, a)$	$a + 2$	$(2a^2 + 6a + 4)/6a$	$(a + 2)/6$	$\frac{(a+2)^2}{a}$
$\mathbb{P}(1, 2, 5)$	8	12/5	4/3	32/5

For $\mathbb{P}(1, 1, a)$, the polytope P is the convex hull of $(1, 0)$, $(0, 1)$ and $(-1, -a)$ and we consider the resolution obtained by adding the ray generated by $(0, -1)$. We use proposition 7.2.1 in order to compute $\widehat{\mu}_P$, with $\nu_1 = 1$, $\nu_2 = \frac{2}{a}$, $\nu_3 = 1$, $\nu_4 = 1$ and $c_1^2(Y) = 8$.

For $\mathbb{P}(1, 2, 5)$, the polytope P is the convex hull of $(1, 0)$, $(0, 1)$ and $(-2, -5)$ and we consider the resolution obtained by adding the rays generated by $(0, -1)$, $(-1, -3)$ and $(-1, -2)$. We use proposition 7.2.1 in order to compute $\widehat{\mu}_P$, with $\nu_1 = 1$, $\nu_2 = \frac{3}{5}$, $\nu_3 = \frac{4}{5}$, $\nu_4 = 1$, $\nu_5 = 1$, $\nu_6 = 1$ and $c_1^2(Y) = 6$.

Notice that in these examples we have $\widehat{\mu}_P = \mu_{P^\circ}$ where μ_{P° is the volume of the polar polytope: this is not a coincidence, see section 8 below.

- The polytope P is not Fano:

Example	μ	$V(\alpha)$	$\mu n/12$	$\widehat{\mu}_P$
$\mathbb{P}(1, \ell, \ell)$	$1 + 2\ell$	$2 + \frac{(\ell-1)(2\ell-1)}{3\ell}$	$(2\ell + 1)/6$	$9 + 2\frac{(\ell-1)^2}{\ell}$
$\mathbb{P}(1, 2, 2, 2)$	7	7	7/4	63/2

For $\mathbb{P}(1, \ell, \ell)$, $\ell \geq 2$, the polytope P is the convex hull of $(1, 0)$, $(0, 1)$ and $(-\ell, -\ell)$. Formula (28) gives

$$\widehat{\mu}_P = c_1(\mathbb{P}^2)c_1(\mathbb{P}^2) + c_1(\mathbb{P}^1)(\ell - 1) - \frac{1}{\ell}c_1(\mathbb{P}^1)(\ell - 1)$$

In this case we have $\widehat{\mu}_P \neq \mu_{P^\circ}$.

For $\mathbb{P}(1, 2, 2, 2)$, P is the convex hull of $(-2, -2, -2)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Formula (28) gives

$$\widehat{\mu}_P = c_1(\mathbb{P}^3)c_2(\mathbb{P}^3) + c_1(\mathbb{P}^2)c_1(\mathbb{P}^2) - \frac{1}{2}c_2(\mathbb{P}^2)$$

7.3.2 Miscellaneous

In order to complete the panorama, let us now consider somewhat different situations:

- let $P_{1,2,2}$ be the polytope with vertices $b_1 = (1, 0)$, $b_2 = (0, 2)$ and $b_3 = (-2, -2)$. Its stacky fan is $\Delta = (N, \Delta, \{b_1, b_2, b_3\})$ where Δ is the fan whose rays are $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -1)$.
- Let $P_{\ell, \ell, \ell}$ be the polytope with vertices $b_1 = (\ell, 0)$, $b_2 = (0, \ell)$ and $b_3 = (-\ell, -\ell)$ where ℓ is a positive integer. Its stacky fan is $\Delta = (N, \Delta, \{b_1, b_2, b_3\})$ where Δ is as above.

We have the following table:

Example	μ	$V(\alpha)$	$\mu n/12$	$\widehat{\mu}_P$
$P_{1,2,2}$	8	3	4/3	10
$P_{\ell, \ell, \ell}$	$3\ell^2$	$(\ell^2 + 3)/2$	$\ell^2/2$	9

This agrees with formula (34)

8 A Noether's formula for two dimensional Fano polytopes

In this section, we still focus on the two dimensional case: P is polytope in $N_{\mathbb{R}} = \mathbb{R}^2$. Recall that $\mu_P := 2 \text{vol}(P)$. Observe first the following: if P is a reflexive polytope, X_{Δ_P} has a crepant resolution and $\widehat{\mu}_P = c_1^2(Y)$ by remark 7.1.2; the anticanonical divisor of Y is nef and $c_1^2(Y) = \mu_{P^\circ}$ by [7, Theorem 13.4.3] so that finally $\widehat{\mu}_P = \mu_{P^\circ}$. Moreover, by corollary 4.2.9, the geometric spectrum of P satisfies $\sum_{i=1}^{\mu_P} (\beta_i - 1)^2 = 2$ and we thus get from corollary 7.2.2 the well-known Noether's formula

$$12 = \mu_P + \mu_{P^\circ} \quad (38)$$

for a reflexive polytope P .

Recall that a convex lattice polytope is *Fano* if the origin is contained in the strict interior of P and if its vertices are primitive lattice points of N , see section 2.3. We have the following generalization of equation (38) for Fano polytopes (a reflexive polytope is Fano):

Theorem 8.0.1 *Assume that P is a full dimensional Fano polytope in $N_{\mathbb{R}}$ with geometric spectrum $\text{Spec}_P^{\text{geo}}(z) = \sum_{i=1}^{\mu_P} z^{\beta_i}$. Then*

$$\sum_{i=1}^{\mu_P} (\beta_i - 1)^2 = \frac{\mu_P}{6} + \frac{\mu_{P^\circ}}{6} \quad (39)$$

where P° is the polar polytope of P .

Proof. Notice first that, because of the Fano assumption, the support function of the \mathbb{Q} -Cartier divisor K_X is equal to the Newton function of P and thus $\rho^*(-K_X) = \sum_{i=1}^r \nu_i D_i$ since $\rho^*(-K_X)$ and $-K_X$ have the same support function. We shall show that

$$\widehat{\mu}_P = \rho^*(-K_X) \rho^*(-K_X) \quad (40)$$

Because $(\rho^*(-K_X))^2 = (-K_X)^2 = \mu_{P^\circ}$ (for the first equality see [7, Lemma 13.4.2] and for the second one see the \mathbb{Q} -Cartier version of [7, Theorem 13.4.3]), equation (39) will follow from theorem 7.1.3. By proposition 7.2.1, we have

$$\widehat{\mu}_P = \sum_{i=1}^r (D_i^2 + \frac{\nu_i}{\nu_{i+1}} + \frac{\nu_{i+1}}{\nu_i}) = \sum_{i=1}^r D_i^2 + \sum_{i=1}^r (\frac{\nu_{i-1}}{\nu_i} + \frac{\nu_{i+1}}{\nu_i})$$

and, as noticed above,

$$\rho^*(-K_X)\rho^*(-K_X) = (\sum_{i=1}^r \nu_i D_i)^2 = \sum_{i=1}^r \nu_i^2 D_i^2 + \sum_{i=1}^r (\nu_i \nu_{i+1} + \nu_i \nu_{i-1})$$

so (40) reads

$$\sum_{i=1}^r (\nu_i^2 - 1) D_i^2 = \sum_{i=1}^r (\nu_i^2 - 1) (-\frac{\nu_{i-1}}{\nu_i} - \frac{\nu_{i+1}}{\nu_i}) \quad (41)$$

Notice the following:

- if v_{i-1} , v_i and v_{i+1} are primitive generators of rays of Y inside a same cone of the fan of X , we have

$$\nu(v_{i-1} + v_{i+1}) = (\frac{\nu_{i-1}}{\nu_i} + \frac{\nu_{i+1}}{\nu_i}) \nu(v_i)$$

because $\nu(v_{i-1}) = \nu_{i-1}$, $\nu(v_i) = \nu_i$ and $\nu(v_{i+1}) = \nu_{i+1}$ and the Newton function is linear on each cone of the fan of X . Because Y is smooth and complete, it follows that

$$v_{i-1} + v_{i+1} = (\frac{\nu_{i-1}}{\nu_i} + \frac{\nu_{i+1}}{\nu_i}) v_i$$

and we get $D_i^2 = -\frac{\nu_{i-1}}{\nu_i} - \frac{\nu_{i+1}}{\nu_i}$.

- otherwise, $\nu_i = 1$ due to the Fano condition.

Equation (41), hence equation (40), follows from these two observations. \square

Example 8.0.2 Let P be the convex hull of $(1,0)$, $(-\lambda_1, -\lambda_2)$ and $(0,1)$ where λ_1 and λ_2 are relatively prime integers: this is the polytope of $\mathbb{P}(1, \lambda_1, \lambda_2)$, see section 6.2. Then,

$$\sum_{i=1}^{\mu_P} (\beta_i - 1)^2 = \frac{1}{6} ((1 + \lambda_1 + \lambda_2) + \frac{(1 + \lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2})$$

Remark 8.0.3 Theorem 8.0.1 is not true if we forget the assumption Fano: for instance, if P is the polytope of $\mathbb{P}(1, \ell, \ell)$, $\ell \geq 2$, we have $\widehat{\mu}_P = 9 + 2(\ell - 1)^2/\ell$ and $\mu_{P^\circ} = (1 + 2\ell)^2/\ell^2$, see examples 7.3. Moreover, it follows from proposition 7.2.1 that $\widehat{\mu}_{\ell P} = \widehat{\mu}_P$ if ℓ is an integer greater or equal than one, and thus $\widehat{\mu}_P$ can't be seen as a volume in general.

Finally, we get the following statement for singularities:

Corollary 8.0.4 Let f be a nondegenerate and convenient Laurent polynomial on $(\mathbb{C}^*)^2$ with spectrum at infinity $\alpha_1, \dots, \alpha_\mu$. Assume that the Newton polytope P of f is Fano. Then

$$\sum_{i=1}^{\mu} (\alpha_i - 1)^2 = \frac{\mu_P}{6} + \frac{\mu_{P^\circ}}{6} \quad (42)$$

where P° is the polar polytope of P . In particular, $\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - 1)^2 \geq \frac{1}{6}$. \square

9 Hertling's conjecture for regular functions

From theorem 7.1.5 we get:

Proposition 9.0.5 *Let f be a convenient and nondegenerate Laurent polynomial with global Milnor number μ and spectrum at infinity $\text{Spec}_f(z) = \sum_{i=1}^{\mu} z^{\alpha_i}$. Let P be its Newton polytope (assumed to be simplicial). Assume that $\widehat{\mu}_P \geq 0$. Then*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 \geq \frac{n}{12} \quad (43)$$

□

In example 7.1.6, corollary 7.2.2, examples 7.3 and corollary 8.0.4 we have $\widehat{\mu}_P \geq 0$ and we expect that it is a general rule. Notice that if true, inequality (43) is the best possible: for instance, in the situation of example 8.0.2 we have

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - 1)^2 = \frac{1}{6} + \frac{(1 + \lambda_1 + \lambda_2)}{6\lambda_1\lambda_2}$$

and the last term on the right can be as small as we want (e.g. $\lambda_1 = p$ and $\lambda_2 = p + 1$ with p large enough). This motivates the following conjecture (recall that $\alpha_1 = 0$ and $\alpha_{\mu} = n$ if f is a convenient and nondegenerate Laurent polynomial, see proposition 5.4.1) which has been already stated without any further comments in [12, Remark 4.15] as a global counterpart of C. Hertling's conjecture for germs of holomorphic functions (see [16], where the equality is inversed). The tameness assumption is discussed in section 5.1.

Conjecture on the variance of the spectrum (global version) *Let f be a regular, tame function on a smooth n -dimensional affine variety U . Then*

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 \geq \frac{1}{12} (\alpha_{\mu} - \alpha_1) \quad (44)$$

where $\alpha_1 \leq \dots \leq \alpha_{\mu}$ is the (ordered) spectrum of f at infinity. □

This is another story, but, as suggested by example 7.1.6, one should expect

$$\frac{1}{\mu} \sum_{i=1}^{\mu} (\alpha_i - \frac{n}{2})^2 = \frac{\alpha_{\mu} - \alpha_1}{12}$$

if f belongs to the ideal generated by its partial derivatives (this is the case for quasi-homogeneous polynomials, see [8] and [16]) because, under mirror symmetry, the multiplication by f on its Jacobi ring corresponds to the cup-product by $c_1(X)$ on the cohomology algebra. Example: $f(x, y) = xy(x - 1)$ for which we have $\mu = 2$ and $\alpha_1 = \alpha_2 = 1$.

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